SURVEY OF CERTAIN BRNO GROUP RESULTS IN THE AREA OF MULTISTRUCTURES OF PREFERENCE RELATIONS

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Abstract: *The contribution contains a certain selection from recently obtained results of Brno group in the area of binary multistructures and hypergroups - of preference relations on general sets of alternatives. As the motivation idea of investigation serves the hypergroup of preferences determined by ternary relation of betweenness in the lattice of relations studied in framework of the group choice theory.*

Key words: Preference relation, semihypergroup, hypergroup.

Preference or "taste" belongs to basic concepts used in the social sciences, particulary in economics, in decision tasks (e.g. in cognitive sciences individual preferences enable choice of objectives - goal) in technical sciences and elsewhere. In particular in microeconomics, preferences of consumers and other entities are modeled with preference relations. Let S be the set of all "packages" of goods and services (or more generally packages "possible worlds"). Then a subset $R \subset S \times S$ (the cartesian square of the set S) is called a preference relation on S if this binary relation R has this property: For a pair of elements $a, b \in S$ we have $a\,R\,b$ or $[a,b] \in R$ if and only if b is at least as preferable as a. More conventional is to say that "b is weakly preferred to a" or just "b is preferred to a". If $a\,R\,b$ but not $b\,R\,a$, then the consumer strictly prefers b to a, which is written $a\,R\,b, a \neq b$. It is to be noted that instead of $a\,R\,b$ mostly it is used the symbol $a \leq b, a \neq b$, stands $a < b$.

The properties possessed by binary relations playing the role of preferences are the following:

1. Reflexivity: The relation $R \subset S \times S$ is reflexive if $a\,R\,a$ for any $a \in S$, i.e. $\Delta_S \subset R$.
2. Transitivity: If $a, b, c \in S$, $a\,R\,b$, $b\,R\,c$ then $a\,R\,c$; equivalently $R \bullet R \subset R$.
3. Completeness: For all $a, b \in S$ we have $a\,R\,b$ or $b\,R\,a$ or both (notice that completeness implies reflexivity). This is also termed as a linearity.

As there has been mentioned above, the present contribution contains some selection from results obtained by informal Brno research group of mathematicians (Jan Chvalina (head), Jaromír Baštinec, Jaroslav Beránek, Ludmila Chvalinová, Jiří Moučka, Michal Novák, Jiřina Novotná, Zdeněk Svoboda, Josef Zapletal) working in the field of algebraic structures of preferences or rather binary relations and their functional transforms in general.

Recall now some basic definitions from the hyperstructure theory, most of which can be found in [5], [6] or [14]. A *hypergrupoid* is a pair $(H, \bullet)$, where $H \neq 0$ and $\bullet : H \times H \to P^*(H)$ is a binary hyperoperation on H. Symbol $P^*(H)$ denotes the system of all nonempty subsets of H. If the associativity axiom $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a, b, c \in H$, then the pair $(H, \bullet)$ is called a *semihypergroup*. If moreover the reproduction axiom $a \bullet H = H = H \bullet a$ for any element $a \in H$ is satisfied , then the pair $(H, \bullet)$ is called a *hypergroup*. A hypergroup $(H, \bullet)$ is called a *transposition hypergroup* or a *join space* if it satisfies the following transposition axiom: For all $a, b, c, d \in H$ the relation $a/b \approx c/d$ implies $a \bullet d \approx b \bullet c$, where $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$. Sets $a/b = \{x \in H; a \in x \bullet b\}$ and $c/d = \{x \in H; c \in x \bullet d\}$ are called *extensions*, or *fractions*. By a *quasi-ordered semigroup* we mean a triple $(G, \bullet, \leq)$, where $(G, \bullet)$ is a semigroup and binary relation $\leq$ is a quasi-ordering (i.e. is reflexive and transitive) on the set G such that for any triple $x, y, z \in G$ with the property $x \leq y$ there holds also $x \bullet z \leq y \bullet z$ and $z \bullet x \leq z \bullet y$. The *principal end*, often called also *principal upper cone* generated by $a \in G$ is a set $[a)_{\leq} = \{x \in G; a \leq x\}$; the *principal lower cone* can be defined dually.

In connection with fuzzy algebras and so-called minimax algebras there are playing an important role *bottleneck algebras*. A bottleneck algebra is defined to be a triple (R, l, u), where R is a totally ordered set, l, u are binary operations on R such that for each pair $a, b \in R$ we have $alb = \min\{a,b\}, aub = \max\{a,b\}$ (cf. paper [24], p. 59).

In [15], Theorem 1 there are characterized bottleneck algebras, i.e. chains of preference relations in terms of seven binary hyperoperations which are naturally defined on ordered sets, particulary lattices of preferences on general sets of alternatives - [15], [17].

Further, let M be a nonempty set and $\precsim$ a preorder, i.e. a reflexive and transitive binary relation, on M. The set M is usually regarded as the universal set of mutually exclusive choice alternatives, while $\precsim$ is called a *preference relation* of an individual or group of individuals over this set. The strict preorder, denoted by $\prec$, is a binary relation on *M* defined by $x \prec y$ if $x \precsim y$ and not $y \precsim x$. As usual, we say that a real function $f{:}M \to R$ is $\precsim$–*increasing* if *f(x)* $\leq$ *f(x)* for every $x, y \in M$ with $x \prec y$. The following definition is adopted from [9], p. 5.

Definition 1. *Let M be a nonempty set and $\prec$ a preorder on M. We say that a nonempty subset U of R^M represents $\precsim$ if*

$$x \precsim y \text{ if and only if } u(x) \leqq u(y) \text{ for all } u \in U$$

for every $x, y \in M$. If such a set U exists, then we say that there exists a multi-utility representation for the preorder (preference relation) $\precsim$. If U is finite (countable), we say that there exists a finite (countable) multi-utility representation for $\precsim$.

It is worth stressing that if a nonempty $U \subseteq R^M$ represents a preorder $\precsim$ on M, then every member of U has to be$\precsim$–increasing but no member of U needs to be strictly $\precsim$–increasing.

Using characteristic functions of principal lower conesgenerated by elements of a preordered set the authors of [9] prove the following:

Proposition 1. ([9], Proposition 1., p. 6).*There exists a multi-utility representation for every preorder.*

Let $C(M)$ be a system of all continuous functions $f{:}M \to R$ considered as an ordered linear space. If $U \subseteq C(M)$, where M is a topological space, we denote by $[U]$ the *exponential-algebraic extension of the system* U, i.e. a set

$$[U] = \{af + bg; f, g \in U, a, b \in R\} \cup \{\exp(af + bg); f, g \in U, a, b \in R\},$$

where $\exp u{:}J \to R$ is the following composed function: $(\exp u)(x) = e^{u(x)}, x \in J$. Notice that for an arbitrary pair $u, v \in [U]$, the notation $u \leqq v$ means that *u(x)* $\leqq$ *v*(x) for any $x \in J$. Using the just mentioned construction we reach the following result:

Theorem 1. *Let $U \subseteq C(J)$ be a continuous multi-utility representation of a continuous preorder on a topological space M of alternatives and $[U]$ be the above defined exponential-algebraic extension of U. If we define a binary hyperoperation*

$$\bullet\ [U] \times [U] \to P([U])$$

by

$$u \bullet v = \bigcup_{[a,b] \in R_0^+ \times R_0^+} \{w;\ w \in [U],\ au + bv \le w\},$$

then $([U],\bullet)$ *is a (commutative) join space. Moreover, if* $[U_i]$ *is the exponential-algebraic extension of a continuous multi-utility representation* U_i *of a continuous preorder* $\precsim_i$, $i = 1,2$, *on the topological space* M *and f:* $([U_1], \leqq_1) \to ([U_2], \leqq_2)$ *is a linear increasing mapping, then* f *is an inclusion homomorphism of the join space* $([U_1],\bullet_1)$ *into the join space* $([U_2],\bullet_2)$.

Increasing bijective transformation of quasi-orderedsets with the operation of composition of mappings are obviously noncommutative groups. However, utility functions with a binary operation of usual addition of functions is a commutative, or abelian, group. Moreover, it is ordered in the usual way, i.e. for any pair of functions $f, g \in G$ we have that $f \le g$ if *f(x)* $\leqq$ *f(y)* for all $x, y \in G$. Results obtained in paper [21] enable us to include a construction of a commutative hypergroup on any ordered abelian group. Indeed, regard an ordered abelian group $(G,+,\le)$ and define a hyperoperation $\bullet : G \times G \to P^*(G)$ in the following way:

$$a \bullet b = \bigcup_{[m,n] \in N_0 \times N_0} [ma + nb)_{\le} = \bigcup_{[m,n] \in N_0 \times N_0} \{x \in G;\ ma + nb \le x\}$$

for any $a, b \in G$. Evidently $(G,\bullet)$ is a commutative hypergroupoid.

Proposition 2. *The above defined hypergroupoid* $(G,\bullet)$ *is an abelian hypergroup.*

In the algebraic theory of hyperstructures there were introduced and studied so-called P–semihypergroups and P–hypergroups (cf. [33] and related papers). The concept is a generalization of the notion of a variant of a semigroup or a sandwich semigroup. In the case of sandwich semigroups of binary relations on a set there exists a close connection to the concept of a relator (cf. [31]). Relators are simply nonvoid collections of reflexive relations on sets. Theory of relators (essentially identical to the generalized uniformities of I. Konishi – 1952 and V. S. Krishnan – 1955) generalizing various uniformities (Á. Császár and R. Z. Domiaty – 1979/80, P. Fletcher and W. F. Lindgren – 1978, 1982) has been intensively studied by Árpád Szász since the end of 1980s in a series of papers and in his monography *Relators, nets and integrals*. We explain the above mentioned concepts on the example of a semigroup of preference relations on a set of some alternatives.

Let R be a fixed binary relation on a set X. If for any two relations A and B in $B(X)$, which is the set of all binary relations on X, we define $A * B$ as ARB, where juxtaposition is the usual composition of relations, we obtain the sandwich semigroup $(B_R(X),*)$ of binary relations on the set X with sandwich relation R (cf. [32]). These semigroups were studied by K. Chase in 1978/9 so that he could apply them in the automata theory. Now, considering a nonempty subset $P \subseteq B(X)$ – in the case that all relations $A \in P$ are reflexive, the set P is called *relator* (cf. [11], [31]) – and defining a binary hyperoperation on $B(X)$ by $A \circ^P B = APB = \{ARB; R \in P\}$ we obtain a P–hypergroup (cf. [33]). Indeed, it is easy to see that for any triad of relations $A, B, C \in B(X)$ there holds

$$A \circ^P (B \circ^P C) = APBPC = \{ASBTC; S,T \in P\} = (A \circ^P B) \circ^P C,$$

i.e. that the so-called *central* P*–hyperoperation* is associative.

The contribution [19] ends with the following remark which is relevant in our further considerations:

Remark 1. *Let us conclude with a remark that the arbitrary set $R(M) \subset P(M \times M)$ of preference relations on a commodity set M can be endowed with the structure of a commutative hypergroup. Being a subset of a power set $P(M \times M)$ of all binary relations on M, the system $R(M)$ of preference relations is naturally partially ordered by set inclusion "$\subseteq$". Then defining*

$$R * S = \{T; T \in R(M), R \subseteq T \text{or} S \subseteq T\}$$

we get easily that the pair $(R(M),*)$ is a commutative hypergroup (cf also [14]).

Let us further denote by $Toc(X)$ a system of commuting tolerances on a set X. Notice that commuting relations have been studied in some papers by Tamás Glavosits and Árpád Szász in e.g. [11], [31] and in particular in the paper *Characterizations of commuting relations* (Institute of Mathematics, Debrecen University – preprint, 9pp.) by Száz, where Theorem 2 states that if R, S are tolerances, i.e. reflexive and symmetric relations, on X, then the following assertions are equivalent:

- **(1)** $RS = SR$,
- **(2)** RS is a tolerance,
- **(3)** $R(x) \cap S(y) \neq \emptyset$ implies $S(x) \cap R(y) \neq \emptyset$ for all $x, y \in X$.

In fact the author of the first result concerning commutativity of equivalences is František Šik, Spisy vyd. Přír. fak. Masarykovy Univ. **3** (1954), 97–102. Some generalizations have been obtained by Ladislav Kosmák in Acta Math. Univ. Comenianae 1980.

In what follows we suppose that P is a subsemigroup of $Toc(X)$. Denote by S the semiring of all even positive integers. Define on $Toc(X)$ a binary relation ρ_P in the following way: For $R, S \in Toc(X)$ we put $[R,S] \in \rho_P$ or simply $R\rho_P S$ whenever there exists an even integer $n \in S$ such that

$$S \in R^{\frac{n}{2}} P R^{\frac{n}{2}} = R^n P = \{R^n T; T \in P\} = PR^n.$$

Proposition 3. *Let $Toc(X)$ be an abelian semigroup of tolerances on X and P its subsemigroup. Then the binary relation $\rho_P \subset Toc(X) \times Toc(X)$ is transitive.*

Further, denoting $\Delta_T = \{[R, R]; R \in Toc(X)\}$ – the identity (or diagonal) relation on the semigroup $Toc(X)$ – we obtained (using Theorem 2.1 from [14]) the following result:

Theorem 2. *Let $Toc(X)$ be an abelian semigroup of tolerance relations on X, P be its subsemigroup and ρ_P be the above defined relation on $Toc(X)$. Denote $\sigma_P = \sigma_P \cup \Delta_T$ and for any pair of tolerances $R, S \in Toc(X)$ define*

$$R * S = \sigma_P(R) \cup \sigma_P(S) = \{T \in Toc(X); R\rho_P T \text{or} S\rho_P T\} \cup \{R, S\}.$$

Then $(Toc(X),*)$ is a commutative, i.e. abelian, hypergroup of tolerances.

In a certain more general approach, let us consider a commutative semigroup S and $\emptyset \neq P \subset S$.

Theorem 3. *Let $(S,\cdot)$ be a commutative monoid (with the unit e) of idempotent elements, i.e. $(S,\cdot)$ is a commutative band. Suppose $P \subset S$ is a submonoid of S (i.e. its carrier set) and $R_P \subset S \times S$ is a binary relation defined by $[x, y] \in R_P$ iff $y \in x \cdot P \cdot x = x \cdot P$. Then the relation*

R_P is reflexive and transitive, i.e. it is a quasi-order on the set X.

As far as the theory of representation of preferences is concerned, continuous representations are also important. Therefore, in a substantial part of the theory of preferences, metric spaces or generally topological spaces serve as structural background.

The paper [20] describes some separation properties of a bitopological space, induced by a reflexive and transitive incomplete preference on an abstract set. Considered bitopological space is endowed with the pair of topologies (the right and the left topologies corresponding to the preference relation in question). There is proved that preferences forming directed graphs without circuits can be characterized in terms of separation axioms equivalently from pairwise T_0 up to quasi-Hausdorff or weakly pairwise T_2. Complete formulation of results can be found in [15], [19], [20].

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