THE PRINCIPLE OF QUOTIENT IN THE COURSE OF DISCRETE MATHEMATICS

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Abstract. The course of Discrete Mathematics is one of the basics of every study program connected to computer science. Due to different high school backgrounds, the students coming to the university possess a very diverse knowledge of combinatorics. Here, we have described an example of a methodological approach resulting in the leveling of student understanding in this area.

Key words: discrete mathematics, quotient principle, combinatorics

Introduction

In the article, we have summarized our experience from teaching Discrete Mathematics (the lectures and the seminars to the lectures) at the University of South Bohemia in the Czech Republic. We are especially interested in the point of view of the students in solving combinatorial word problems (see the classical book [7], for example). These problems are very often supposed to be non-standard tasks and the students think that one can solve them only if he or she only knows some special ‘tricks’.

Our students were surprised when it was explained to them that combinatorics is a legitimate mathematical discipline with its own terminology, theoretical background, general theorems and formulas, and its own standard procedures; (the existence of the handbook [2], [3] was a great surprise to many of them). Here, we have demonstrated a piece of our successful approach.

Material and methods

Throughout the paper, we work with finite sets. If X is a set then we denote the size of X by $|X|$, the set of all ordered pairs $[a, b]$ of elements of X by $X \times X$ (*the cartesian square* of X), the set of all subsets of X by $\boldsymbol{P}(X)$ (*the power set* of X), and the set of all subsets of X of size equal to k by $\boldsymbol{P}_k(X)$. The system $\{ X_j \ ; j \in J \}$ of nonempty subsets of X is called a partition of X, if it is pair-wise disjoint and their union equals X. Moreover, if the size of all partition classes equals k we talk about k-partition.

The Quotient Principle. If $\{ X_j \ ; j \in J \}$ is a k-partition of X into | J | nonempty classes. Then

$$|J| = \frac{|X|}{k}$$

The proof is evident because X is divided into $|J|$ parts of the same size k, i.e. $|X| = |J| \cdot k$.

For a mapping, we use the standard notation $f\colon X \to Y$; $f(a)$ is the image of element $a \in X$, $f(S)$ is the image of subset $S \subseteq X$. The *composite mapping* of two mappings f and g (if is exists) is denoted by $g \circ f$ and it works as follows: $(g \circ f)(a) = g(f(a))$. If mapping $f : X \to Y$ is a one-to-one correspondence we call it *a bijection.* If both X and Y have the size of n then the number of bijections from X to Y equals $n!$. The composition of two bijections is a bijection as well. The inverse f^{-1} of a bijection f is a bijection, too. We deal with four kinds of *objects* (see [1], for example):

a colored set $O = \langle X, \chi \rangle$ where $\chi : X \to C$ is a mapping (C is called *the set of colors*, the image $\chi(a)$ is called *the color of element a* in object O), *a directed graph* $O = \langle X, R \rangle$ where R is a subset of $X \times X$ (the elements of R have the form of $[a, b]$ and are called *arrows*) ,

an undirected graph $O = \langle X, R \rangle$ where R is a subset of $\boldsymbol{P}_2(X)$ (the elements of R have the form of $\{a, b\}$ and are called *edges*),

a partition $O = \langle X, \mathrm{R} \rangle$ where R is a partition of X (the elements of R are subsets of X and they are called *partition classes*) .

Definition. Let O be an object on X and let $f : X \to Y$ be a bijection. The *f-copy* of object O denoted by $f(O)$ is the object on Y of the same kind as O described as follows:

if $O = \langle X, \chi \rangle$ is a colored set with coloring $\chi : X \to C$ then $f(O) = \langle Y, \chi \circ f^{-1} \rangle$ where $\chi \circ f^{-1} : Y \to C$ is the coloring of $f(O)$,

if $O = \langle X, \mathrm{R} \rangle$ is a directed graph then $f(O) = \langle X, R_f \rangle$ where R_f consists of all arrows of the form $[f(a), f(b)]$ provided $[a, b] \in R$,

if $O = \langle X, \mathrm{R} \rangle$ is an undirected graph then $f(O) = \langle X, R_f \rangle$ where R_f consists of all edges of the form $\{f(a), f(b)\}$ provided $\{a, b\} \in R$,

if $O = \langle X, \mathrm{R} \rangle$ is a partition then $f(O) = \langle X, R_f \rangle$ where R_f consists of all partition classes of the form $f(X_j)$ provided X_j is a partition class of O.

If $O_2 = f(O_1)$ we say that O_1 and O_2 are isomorphic *objects* and f is called an *isomorphism*. An isomorphism of an object O to itself is called an automorphism and we denote by aut*(O)* the number of automorphisms of O.

Main Theorem. Let X and Y be two sets of the same size n, let O be an object on X. If the number of distinct copies of O on Y is denoted by copy(O), then

$$\mathrm{copy}(O) = \frac{n\,!}{\mathrm{aut}(O)}$$

Outline of proof (see [5] for more details). Let G be the set of all automorphisms of O (it means that $|G| = \mathrm{aut}(O)$), and let F be the set of all bijections $f : X \to Y$ (it means that $|F| = n!$). The relation ~ on F is defined as follows: $f_1 \sim f_2$ if and only if $f_1(O) = f_2(O)$. It is obvious that ~ is reflexive, symmetric, and transitive; thus it is an equivalence relation which defines a partition of F into equivalence classes $\{ F_j \,;\, j \in J \}$. The number of classes $|\,\mathrm{J}\,|$ equals the number of distinct copies of O on Y, i.e. $\mathrm{copy}(O) = |\,\mathrm{J}\,|$. Choose a class F_j and a bijection $f_0 \in F_j$. Now, define mapping $\beta : F_j \to G$ as $\beta(f) = f_0^{-1} \circ f$. It follows immediately from the definition of β and from the choice of f_0 that β is a bijection and therefore $|F_j| = |G| = \mathrm{aut}(O)$ for every $j \in J$. We apply The Quotient Principle on k-partition $\{ F_j \,;\, j \in J \}$ of F with $k = \mathrm{aut}(O)$ which yields:

$$\mathrm{copy}(O) = |\,J\,| = \frac{|\,F\,|}{k} = \frac{n\,!}{\mathrm{aut}(O)}$$

Note. The proof shows that the number copy(O) does not depend on the choice of set Y.

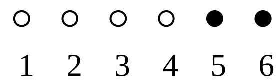
Results and discussion

On 4 examples of counting problems, we demonstrate the use of The Quotient Principle in the form of the Main Theorem emphasizing its generality. Since it always requires determining the value of aut*(O)* for a specific object O we concentrate on this problem. Also, we use *The Multiplication Counting Principle* which, in fact, says that $|\,X \times Y\,| = |\,X\,| \cdot |\,Y\,|$ (holding for more than 2 sets as well). Below, the notation $\mathrm{I}[n] = \{1, 2, \ldots, n\}$ for the intervals of the first n natural numbers is frequently exploited.

Example 1. *Explaining the meaning of binomial coefficients.*
Chosen m, n, $0 \le m \le n$. The object will be colored set $O = \langle \mathrm{I}[n], \chi \rangle$ where $\chi(a)$ = *white* for $a \le m$, and $\chi(a)$ = *black* for $a > m$. Thus, $\mathrm{I}[n]$ is divided into two unicolor parts, the ‘white’ and the ‘black’ one, i.e. $\mathrm{I}[n] = W \cup B$, $|W| = m$ (the white color chooses the m-element subset $S = \{1,2, \dots, m\}$ of $\mathrm{I}[n]$). Every automorphism f of O must preserve the colors. Thus, $f(W) = W$ (there are $m!$ such partial bijections), and $f(B) = B$ (there are $(n-m)!$ such partial bijections). Using the Multiplication Counting Principle we obtain $\mathrm{aut}(O) = m! \cdot (n-m)!$ and finally

$$\mathrm{copy}(O) = \frac{n!}{\mathrm{aut}(O)} = \frac{n!}{m! \cdot (n-m)!} = \binom{n}{m}$$

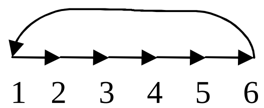
It means that every copy of O on an n-element set Y chooses some m-element subset of Y and the number of copies is the size of $\boldsymbol{P}_m(Y)$. We have shown that the above binomial coefficient calculates the number of m-element subsets of any n-element set Y.

The picture on the right shows the case $n = 6$ and $m = 4$ (4 white elements and 2 black elements in I[6] are chosen). All 4-element subsets in any n-element set Y are obtained as copies of this object and the number of them is $\binom{6}{4} = 15$.	

Example 2 (Combinatorics). *The number of cyclic permutations.*
Chosen $n \ge 2$. The object will be directed graph $O = \langle \mathrm{I}[n], R \rangle$ where $R = \{[i, i+1]; i < n\} \cup \{[n, 1]\}$. Now, every automorphism f of O is uniquely determined by the image of $f(1)$ which has n different possibilities, so $\mathrm{aut}(O) = n$. Using The Main Theorem we obtain

$$\mathrm{copy}(O) = \frac{n!}{\mathrm{aut}(O)} = \frac{n!}{n} = (n-1)!\,.$$

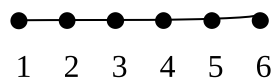
It means that every copy of O on an n-element set Y determinates one cyclic permutation on Y. We have shown that the number of distinct cyclic permutations of n elements is $(n-1)!$.

The picture on the right shows the case $n = 6$, i.e. the standard cyclic permutation on I[6]. The number of distinct cyclic permutations of 6 elements is $(6-1)! = 5! = 120$.	

Example 3 (Graph Theory). *The number of Hamiltonian paths in the complete graph*
Chosen $n \ge 2$. The object will be undirected graph $O = \langle \mathrm{I}[n], R \rangle$ where $R = \{\{i, i+1\}; i < n\}$. Now, every automorphism f of O is uniquely determined by the image of $f(1)$ which has 2 different possibilities, namely $f(1) = 1$ or $f(1) = n$. Using The Main Theorem we get

$$\mathrm{copy}(O) = \frac{n!}{\mathrm{aut}(O)} = \frac{n!}{2}\,.$$

It means that every copy of O on an n-element set Y determines one Hamiltonian path in the complete graph on Y. The number of distinct Hamiltonian paths on n elements is $n!/2$.

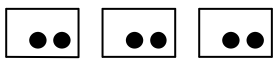
The picture on the right shows the case $n = 6$. The standard path on I[6] has 5 edges forming a ‘chain’. The number of distinct Hamiltonian paths on 6 elements $6!/2 = 360$.	

Example 4 (Combinatorics). *The number of regular partitions*

Chosen $m \geq 1$, and $n = 2m$. We put $J = \mathrm{I}[m]$ and for every $j \in J$ let $X_j = \{2j-1, 2j\}$. The object will be 2-partition $O = \langle \mathrm{I}[n], R \rangle$ where the system $R = \{\{2j, 2j-1\};\ 1 \leq j \leq m\}$ consists of m classes of size 2. Every automorphism f of O must preserve the partition classes and therefore it determines a bijection $f^* : J \to J$ such that for every $j \in J$ there is $f(X_j) = X_{f^*(j)}$ (there are 2! = 2 such partial bijections for every $j \in J$). It is clear that for each bijection $g : J \to J$ there are exactly 2^m automorphisms f of O such that $f^* = g$. Because there are $|J|! = m!$ bijections $g : J \to J$, we conclude that $\mathrm{aut}(O) = m! \cdot 2^m$ (the Multiplication Counting Principle). Using the Main Theorem we get

$$\mathrm{copy}(O) = \frac{n!}{\mathrm{aut}(O)} = \frac{(2m)!}{m! \cdot 2^m}.$$

We have derived the formula calculating the number of distinct 2-partitions.

The picture on the right shows the case $m = 3$ and $n = 6$ with the standard 2-partition on I[6]. The number of distinct 2-partitions on 6 elements is $\frac{(6)!}{3! \cdot 2^3} = 15$.	 1 2 3 4 5 6

Conclusion

We have given our students a tool which helps them in solving different kinds of combinatorial word problems. The main advantage is that the procedures used are standard and avoid the need of some special ‘tricks’. Moreover, mastering the technique of automorphism counting is a good preparation for understanding the theory of transformation groups.

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